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Cyclic vectors of diagonal operators on the space of entire functions

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Abstract

The purpose of this paper is to study cyclic vectors and invariant subspaces of operators on the space of entire functions having as eigenvectors the monomials z^n.

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1. Introduction

A vector x in a complete metrizable topological vector space $\mathcal{X}$ is said to be *cyclic* for a continuous linear operator $T : \mathcal{X} \to \mathcal{X}$ on $\mathcal{X}$ if the closed linear span of the orbit $\{T^n x \colon n \geqslant 0\}$ of x under T is all of $\mathcal{X}$. Operators which have a cyclic vector are said to be *cyclic*. Cyclicity results yield interesting approximation results. For instance, the Weierstrass Approximation Theorem asserts that the function $f(x) \equiv 1$ on $[0, 1]$ is cyclic

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for the operator $T: g(x) \to xg(x)$ of multiplication by x on the Banach space $C([0, 1])$ of continuous functions on $[0, 1]$.

A closed subspace $\mathcal{M}$ of $\mathcal{X}$ is *invariant* for $T: \mathcal{X} \to \mathcal{X}$ if $Tx \in \mathcal{M}$ for all $x \in \mathcal{M}$. The closed linear span of the orbit of any vector x under T is the smallest closed invariant subspace for T containing x. Hence, a vector x is cyclic for T if and only if the smallest closed invariant subspace for T containing x is all of $\mathcal{X}$. The importance of cyclic vectors derives from the long standing study of invariant subspaces of operators and the approximation results they yield.

Cyclic vectors and invariant subspaces of operators on a Hilbert space which are diagonalizable with respect to an orthonormal basis have been well studied (see, for instance, Wermer [24], Brown et al. [3], Sarason [18,19], Scroggs [21], Sibilev [23], and Nikol'skiĭ [15]). The purpose of this paper is to study cyclic vectors and invariant subspaces of the analogous class of operators on the space of entire functions. The preliminaries are as follows.

We denote by $\mathcal{H}(\mathbf{C})$ the vector space of functions analytic on the entire complex plane $\mathbf{C}$. It follows from the Radius of Convergence Formula that a function $f(z) \equiv \sum_{n=0}^{\infty} a_n z^n$ is in $\mathcal{H}(\mathbf{C})$ if and only if $\limsup |a_n|^{1/n} = 0$. When endowed with the topology of uniform convergence on compacta, the space $\mathcal{H}(\mathbf{C})$ is an example of a complete locally convex topological vector space. Moreover, the topology of $\mathcal{H}(\mathbf{C})$ is induced by the invariant metric $\rho(f, g)$ defined by $\rho(f, g) \equiv \sum_{n=0}^{\infty} \|f - g\|_n / \{2^n(1 + \|f - g\|_n)\}$ where here $\|h\|_n \equiv \sup\{|h(z)|\colon |z| \leqslant n\}$ for all functions h in $\mathcal{H}(\mathbf{C})$ and all $n \geqslant 0$ (see Rudin [17]).

Any linear map D on $\mathcal{H}(\mathbf{C})$ having as eigenvectors the monomials z^n with associated eigenvalues λ_n is given formally by $D: \sum_{n=0}^{\infty} a_n z^n \to \sum_{n=0}^{\infty} \lambda_n a_n z^n$. The linear map D defines a continuous linear operator on all of $\mathcal{H}(\mathbf{C})$ if and only if $\limsup |\lambda_n|^{1/n} < \infty$ (see Lemma 1). In this paper, any operator D on $\mathcal{H}(\mathbf{C})$ for which there exists a sequence of complex numbers $\{\lambda_n\colon n \geqslant 0\}$ with $\limsup |\lambda_n|^{1/n} < \infty$ and $D(z^n) = \lambda_n z^n$ for all $n \geqslant 0$ will be called *a diagonal operator on* $\mathcal{H}(\mathbf{C})$ *having eigenvalues* $\{\lambda_n\}$. The purpose of this paper is to study cyclic vectors and invariant subspaces of diagonal operators on $\mathcal{H}(\mathbf{C})$. Of particular interest will be conditions on the eigenvalues $\{\lambda_n\}$ of a diagonal operator D on $\mathcal{H}(\mathbf{C})$ for D to admit *spectral synthesis*; that is, for every closed invariant subspace $\mathcal{M}$ for D to equal the closed linear span of the eigenvectors for D which are in $\mathcal{M}$.

In Section 2, we show that a diagonal operator D on $\mathcal{H}(\mathbf{C})$ is cyclic if and only if the eigenvalues of D are distinct. In this case, we show D has a dense set of cyclic vectors.

In Section 3, we give equivalent conditions for a cyclic diagonal operator on $\mathcal{H}(\mathbf{C})$ to admit spectral synthesis and show that every cyclic diagonal operator whose eigenvalues are bounded admits spectral synthesis.

In Section 4, we show that the uncountable collection of diagonal operators on $\mathcal{H}(\mathbf{C})$ each of whose set of eigenvalues are separated has a dense set of common cyclic vectors.

Throughout this paper, we will apply the following results concerning the space of entire functions $\mathcal{H}(\mathbf{C})$ and its dual without further reference (see Rudin [17], Levin [10], or the papers [7,8] of Iyer). A linear map $L: \mathcal{H}(\mathbf{C}) \to \mathbf{C}$ is continuous if and only if there exists a sequence $\{l_n\}$ of complex numbers for which $\sup |l_n|^{1/n} < \infty$ and $L(\sum_{n=0}^{\infty} c_n z^n) = \sum_{n=0}^{\infty} l_n c_n$ for every function $\sum_{n=0}^{\infty} c_n z^n$ in $\mathcal{H}(\mathbf{C})$. Moreover, a closed subspace $\mathcal{M}$ of $\mathcal{H}(\mathbf{C})$ is not all of $\mathcal{H}(\mathbf{C})$ if and only if there exists a nonzero continuous functional $L : \mathcal{H}(\mathbf{C}) \to \mathbf{C}$ for which $L(x) \equiv 0$ for all $x \in \mathcal{M}$. Finally, the closure of any subset of $\mathcal{H}(\mathbf{C})$ and the weak closure of the subset coincide.

2. Cyclicity results

In this section, we show that a diagonal operator D on $\mathcal{H}(\mathbf{C})$ is cyclic if and only if the eigenvalues of D are distinct. In this case, we show D has a dense set of cyclic vectors.

We begin by showing that a linear map D having eigenvectors z^n with associated eigenvalues λ_n is continuous on $\mathcal{H}(\mathbf{C})$ if and only if $\limsup |\lambda_n|^{1/n} < \infty$ (also see [7]).

Lemma 1. *Let $\{\lambda_n\}$ be any sequence of complex numbers. Then $D : \sum_{n=0}^{\infty} a_n z^n \mapsto \sum_{n=0}^{\infty} \lambda_n a_n z^n$ defines a continuous linear map from $\mathcal{H}(\mathbf{C})$ into $\mathcal{H}(\mathbf{C})$ if and only if $\limsup |\lambda_n|^{1/n} < \infty$.*

Proof. If $\sum_{n=0}^{\infty} \lambda_n a_n z^n$ is in $\mathcal{H}(\mathbf{C})$ whenever $\sum_{n=0}^{\infty} a_n z^n$ is in $\mathcal{H}(\mathbf{C})$, then we have $\limsup |\lambda_n a_n|^{1/n} = 0$ whenever $\limsup |a_n|^{1/n} = 0$ and so $\limsup |\lambda_n|^{1/n} < \infty$. Conversely, if $\limsup |\lambda_n|^{1/n} < \infty$, then $D : \sum_{n=0}^{\infty} a_n z^n \mapsto \sum_{n=0}^{\infty} \lambda_n a_n z^n$ defines a linear map from $\mathcal{H}(\mathbf{C})$ into $\mathcal{H}(\mathbf{C})$. It follows from the Closed Graph theorem (see [17, Theorem 2.15, p. 51]) that D is continuous. □

We now derive a simple test for an entire function to be cyclic for a diagonal operator on $\mathcal{H}(\mathbf{C})$.

Proposition 2. *Let D be a diagonal operator on $\mathcal{H}(\mathbf{C})$ having eigenvalues $\{\lambda_n\}$ and let $f(z) \equiv \sum_{n=0}^{\infty} a_n z^n$ be any entire function. The following are equivalent:*

(i) *f fails to be cyclic for D,*
(ii) *the closed linear span of the orbit $\{\sum_{n=0}^{\infty} a_n \lambda_n^k z^n\colon k \geqslant 0\}$ of f under D is not all of $H(\mathbf{C})$, and*
(iii) *there exists a sequence $\{l_n\}$ of complex numbers, not all zero, for which*

$$\sup |l_n|^{1/n} < \infty \quad \text{and} \quad 0 \equiv \sum_{n=0}^{\infty} l_n a_n \lambda_n^k \quad \text{for all } k \geqslant 0.$$

The proof of Proposition 2 follows immediately from the topological vector space version of the Hahn–Banach theorem.

As an immediate consequence of the preceding lemma, we have that a simple necessary condition for an entire function $\sum_{n=0}^{\infty} a_n z^n$ to be cyclic for a diagonal operator D is that $a_n \neq 0$ for all $n \geqslant 0$. It also follows that a simple necessary condition for a diagonal operator D to be cyclic is that the eigenvalues $\{\lambda_n\}$ of D be distinct (that is, that $\lambda_m \neq \lambda_n$ whenever $m \neq n$). We prove the converse, namely, that a diagonal operator is cyclic whenever its eigenvalues are distinct.

Proposition 3. *Let D be a diagonal operator on $\mathcal{H}(\mathbf{C})$ having eigenvalues $\{\lambda_n\}$. Then D is cyclic if and only if $\lambda_m \neq \lambda_n$ whenever $m \neq n$. In this case, D has a dense set of cyclic vectors.*

Proof. We have already observed that in order for D to be cyclic, the eigenvalues of D must be distinct. Conversely, suppose that $\lambda_m \neq \lambda_n$ whenever $m \neq n$. Since D is continuous, $K \equiv \limsup |\lambda_r|^{1/r} < \infty$, and so $|\lambda_r| \leqslant (K+1)^r$ for all r sufficiently large. Hence there exists a constant $c \geqslant 1$ such that $|\lambda_r| \leqslant c(K+1)^r$ for all $r \geqslant 0$. For each $n \geqslant 1$, define $\alpha_n \equiv \min\{|\lambda_k - \lambda_i|\colon i, k \leqslant n;\ i \neq k\}$ and $\beta_n \equiv \min\{1, \alpha_n\}$.

For each positive integer n and each k in $\{0, 1, 2, \ldots, n\}$, the polynomial

$$p_{n,k}(z) \equiv \prod_{i=0,\, i\neq k}^{n} \frac{z - \lambda_i}{\lambda_k - \lambda_i}$$

is well defined since the eigenvalues of D are distinct. Since the sequence $\{\beta_n\}$ is decreasing, and $\beta_n \leqslant 1$ and $\beta_n \leqslant \alpha_n$ for all n, we have for all $r > n$ that

$$\left|p_{n,k}(\lambda_r)\right| \leqslant \prod_{i=0,\, i\neq k}^{n} \frac{2c(K+1)^r}{\alpha_n} \leqslant \left\{\frac{2c(K+1)^r}{\beta_n}\right\}^n \leqslant \left\{\frac{2c}{\beta_r}\right\}^r (K+1)^{nr}.$$

We show that the set $\mathcal{C}$ of entire functions $\sum_{n=0}^{\infty} a_n z^n$ for which there exists a constant α for which $0 < |a_r| \leqslant \alpha\beta_r^r/\{(2c)^r r^{r^r}\}$ for all $r \geqslant 0$ is a dense set of cyclic vectors for D. Let $f_0(z) \equiv \sum_{k=0}^{\infty} a_k z^k$ be an arbitrary function in $\mathcal{C}$. We show that f_0 is cyclic for D. To this end, let L be an arbitrary functional on $\mathcal{H}(\mathbf{C})$. So there exists a sequence $\{l_n\}$ of complex numbers for which $B \equiv \sup |l_n|^{1/n} < \infty$ and $L(\sum_{n=0}^{\infty} c_n z^n) = \sum_{n=0}^{\infty} l_n c_n$ for all functions $\sum_{n=0}^{\infty} c_n z^n$ in $\mathcal{H}(\mathbf{C})$.

Let k be any nonnegative integer. We have that $p_{n,k}(\lambda_r) = 0$ for all $r \leqslant n$ with $r \neq k$, and that $p_{n,k}(\lambda_k) = 1$ for all n with $k \leqslant n$. Hence for any integer n greater than $2B(K+1)$, we have that $\{B(K+1)^n/n^n\} \leqslant 1/2^n$ and so

$$\left|L\left(p_{n,k}(D) f_0 - a_k z^k\right)\right| = \left|\sum_{r>n} p_{n,k}(\lambda_r) a_r l_r\right| \leqslant \sum_{r>n} \left\{\frac{2c}{\beta_r}\right\}^r (K+1)^{nr} \frac{\alpha\beta_r^r}{(2c)^r r^{r^r}} B^r$$
$$\leqslant \alpha \sum_{r>n} \left\{\frac{B(K+1)^n}{n^n}\right\}^r \leqslant \alpha \sum_{r>n} \frac{1}{2^{rn}}$$

which tends to zero as n tends to infinity. Since L is an arbitrary functional on $\mathcal{H}(\mathbf{C})$, $a_k z^k$ is in the weak closure of the linear span of the orbit of f_0 under D. Since f_0 is in $\mathcal{C}$, $a_k \neq 0$ and so z^k is in the weak closure of the linear span of the orbit of f_0 under D for all $k \geqslant 0$. Since the weak closure of the linear span of the orbit of f_0 under D and the closure of the linear span of the orbit of f_0 under D in $\mathcal{H}(\mathbf{C})$ coincide and the monomials have dense linear span in $\mathcal{H}(\mathbf{C})$, it follows that f_0 is cyclic for D.

Let $g(z) \equiv \sum a_k z^k$ be an arbitrary function in $\mathcal{H}(\mathbf{C})$. Since g is entire, we have that $\limsup |a_k|^{1/k} = 0$, and so for each positive integer m there exists a positive integer $N(m)$ for which $|b_k| \leqslant 1/m^k$ for all $k \geqslant N(m)$. Without loss of generality, $N(m) \geqslant m$. Define $g_m(z) \equiv \sum a_{m,k} z^k$ where $a_{m,k} \equiv a_k$ for $k \leqslant N(m)$ and $a_{m,k} \equiv 1/k^{k^k}$ for $k > N(m)$. The sequence $\{g_m\}$ converges weakly to g and so the weak closure of $\mathcal{C}$ is all of $\mathcal{H}(\mathbf{C})$. However, the weak closure of $\mathcal{C}$ and the closure of $\mathcal{C}$ in $\mathcal{H}(\mathbf{C})$ coincide and so $\mathcal{C}$ is dense in $\mathcal{H}(\mathbf{C})$. The result follows. □

The rate of decay of the coefficients defining the collection $\mathcal{C}$ of entire functions in the preceding proof may be improved significantly.

There is an extensive literature concerning sets of entire functions whose closed linear spans equal all of $\mathcal{H}(\mathbf{C})$. For instance, the set of translates $\{f(z+\lambda_n)\}$ of an entire function f was studied by Iyer (see [7, Theorem 4, p. 877]) and Boas [2]; the set of derivatives $\{f^{(n)}(z)\}$ of an entire function was studied by Iyer (see [7, Theorem 4, p. 877] and [9]); and the set of dilations $\{f(\lambda_n z)\}$ of an entire function was studied by Iyer (see [8, Theorem 2, p. 875]), Boas [2], Gelfand [4], and Markushevich [12], among others. Iyer also showed that if f is an entire function, then the closed linear span of $\{z^n f(z)\}$ is all of $\mathcal{H}(\mathbf{C})$ if and only if $f(z) \neq 0$ for all $z \in \mathbf{C}$ (see Iyer [8]). Nagnibida showed more generally that if f and g are entire functions, then the closed linear span of $\{f^n(z)g(z)\}$ is all of $\mathcal{H}(\mathbf{C})$ if and only if $f(z) = \alpha z + \beta$ where $\alpha \neq 0$ and $g(z) \neq 0$ for all $z \in \mathbf{C}$ (see Nagnibida [13]).

Orbits $\{\sum_{n=0}^{\infty} a_n \lambda_n^k z^n\colon\ k \geqslant 0\}$ of entire functions $\sum_{n=0}^{\infty} a_n z^n$ under the action of diagonal operators D having eigenvalues $\{\lambda_n\}$ fall under this purvue and are discussed in Proposition 2 which thus provides a connection between the classical literature and the study of cyclic vectors of diagonal operators on $\mathcal{H}(\mathbf{C})$. See, for instance, Theorem 3 of Iyer [7, p. 876]. Also see the paper of Schwartz [20] on mean periodic functions under whose purvue much of this paper falls.

It is also possible for the orbit $\{T^n f\}$ of an entire function f under the action of a continuous linear operator T on $\mathcal{H}(\mathbf{C})$ (without taking the linear span) to have closure equalling all of $\mathcal{H}(\mathbf{C})$. Any operator for which such an entire function exists is said to be *hypercyclic*. The operators of differentiation $f \to f'$ and translation $f(z) \to f(z+1)$ on $\mathcal{H}(\mathbf{C})$, for instance, are hypercyclic (see Birkhoff [1] and MacLane [11]). For a survey of hypercyclicity, see Grosse-Erdmann [5].

3. Spectral synthesis

A continuous linear operator $T : \mathcal{X} \to \mathcal{X}$ on a complete metrizable topological vector space $\mathcal{X}$ is said to *admit spectral synthesis* if every closed invariant subspace $\mathcal{M}$ for T equals the closed linear span of the eigenvectors for T contained in $\mathcal{M}$. By definition, a diagonal operator on $\mathcal{H}(\mathbf{C})$ having eigenvalues $\{\lambda_n\}$ has as eigenvectors the monomials z^n. If D is cyclic, then the eigenvalues are distinct and the monomials are the only eigenvectors for D. Hence a cyclic diagonal operator on $\mathcal{H}(\mathbf{C})$ admits spectral synthesis if and only if the lattice of closed invariant subspaces of D consists precisely of the closed linear span of sets $\{z^n\colon\ n \in N\}$ of monomials where N is an arbitrary subset of nonnegative integers.

Proposition 5 gives various equivalent conditions for a cyclic diagonal operator on $\mathcal{H}(\mathbf{C})$ to admit spectral synthesis. Recall that a simple necessary condition for an entire function $\sum_{n=0}^{\infty} a_n z^n$ to be cyclic for a cyclic diagonal operator D on $\mathcal{H}(\mathbf{C})$ is that $a_n \neq 0$ for all $n \geqslant 0$. The result shows, for instance, that the converse holds only for those cyclic diagonal operators D on $\mathcal{H}(\mathbf{C})$ admitting spectral synthesis.

We begin with the following technical lemma.

Lemma 4. *Let $\mathcal{M}$ be any closed subspace of $\mathcal{H}(\mathbf{C})$ other than the whole space $\mathcal{H}(\mathbf{C})$ or $\{0\}$ and define K to be the set of nonnegative integers k for which there exists a function $\sum_{n=0}^{\infty} a_n z^n$ in $\mathcal{M}$ with $a_k \neq 0$. Then there exists a function $\sum_{n=0}^{\infty} a_n z^n$ in $\mathcal{M}$ for which $a_k \neq 0$ for all k in K.*

Proof. By means of contradiction, suppose that no such function in $\mathcal{M}$ exists. Then $\mathcal{M} = \bigcup_{k\in K} \mathcal{M}_k$ where $\mathcal{M}_k \equiv \{h(z) \equiv \sum_{r=0}^{\infty} a_r z^r \in \mathcal{M}\colon\ a_k = 0\}$ for each k in K. Since $\mathcal{M}$ is closed in $\mathcal{H}(\mathbf{C})$, it is complete, and hence of second category in $\mathcal{M}$. In order to obtain a contradiction to the Baire Category Theorem, we need only show that $\mathcal{M}_k$ is of first category in $\mathcal{M}$ for each k in K. To this end, let k be any nonnegative integer in K. We show that $\mathcal{M}_k$ is, in fact, nowhere dense in $\mathcal{M}$. By means of contradiction, suppose that $\mathcal{M}_k$ is not nowhere dense. Since $\mathcal{M}_k$ is closed in $\mathcal{M}$, it must be the case that the interior $(\mathcal{M}_k)^\circ$ of $\mathcal{M}_k$ in $\mathcal{M}$ is nonempty. Hence there exists a nonempty open set Θ in $\mathcal{H}(\mathbf{C})$ for which $(\mathcal{M}_k)^\circ = \Theta \cap \mathcal{M}$.

Let h be any function in $(\mathcal{M}_k)^\circ = \Theta \cap \mathcal{M}$. Since k is in K, there exists a function $f_k(z) \equiv \sum_{r=0}^{\infty} a_r z^r$ in $\mathcal{M}$ for which $a_k \neq 0$. Since Θ is open, there exists a positive number ϵ for which the open ball $B(h,\epsilon)$ in $\mathcal{H}(\mathbf{C})$ with center h and radius ϵ is a subset of Θ. We show that $h + cf_k$ is in Θ whenever c is sufficiently small. For any function g in $\mathcal{H}(\mathbf{C})$ and any nonnegative integer i, we denote $\|g\|_i \equiv \sup\{|g(z)|\colon\ |z| \leqslant i\}$. By the Maximum Modulus Principle, we have that $\|g\|_i \leqslant \|g\|_j$ whenever $i \leqslant j$. Let N be any positive integer for which $\sum_{i=N+1}^{\infty} 1/2^i < \epsilon/2$. For any constant c with $0 < c < \epsilon/(2\|f_k\|_N)$, we have that the distance between h and $h + cf_k$ in $\mathcal{H}(\mathbf{C})$ is

$$
\begin{aligned}
\rho(h, h+cf_k) &= \sum_{i=0}^{\infty} \frac{1}{2^i} \frac{\|cf_k\|_i}{1+\|cf_k\|_i} = \sum_{i=0}^{N} \frac{1}{2^i} \frac{\|cf_k\|_i}{1+\|cf_k\|_i} + \sum_{i=N+1}^{\infty} \frac{1}{2^i} \frac{\|cf_k\|_i}{1+\|cf_k\|_i} \\
&\leqslant \sum_{i=0}^{N} \frac{1}{2^i} \|cf_k\|_i + \sum_{i=N+1}^{\infty} \frac{1}{2^i} \leqslant c\|f_k\|_N + \epsilon/2 < \epsilon.
\end{aligned}
$$

Hence $h + cf_k$ is in $B(h,\epsilon) \subseteq \Theta$ whenever c is in $(0, \epsilon/(2\|f_k\|_N))$. Since $f_k(z) = \sum_{r=0}^{\infty} a_r z^r$ where $a_k \neq 0$, there exists a constant c in $(0, \epsilon/(2\|f_k\|_N))$ with $h(z) + cf_k(z) \equiv \sum_{r=0}^{\infty} b_r z^r$ where $b_k \neq 0$. Since h is in $(\mathcal{M}_k)^\circ = \Theta \cap \mathcal{M} \subseteq \mathcal{M}$ and f_k is in $\mathcal{M}$, we have that $h + cf_k$ is in $\mathcal{M}$. Moreover, $h + cf_k$ is in Θ. Hence $h + cf_k$ is in $\Theta \cap \mathcal{M} = (\mathcal{M}_k)^\circ \subseteq \mathcal{M}_k$. Hence by definition of $\mathcal{M}_k$, we have that $b_k = 0$, a contradiction. That is, $(\mathcal{M}_k)^\circ$ is empty and so $\mathcal{M}_k$ is nowhere dense. The result follows. □

Proposition 5. *Let D be any cyclic diagonal operator on $\mathcal{H}(\mathbf{C})$ having distinct eigenvalues $\{\lambda_n\}$. Then the following are equivalent*:

- (i) *D admits spectral synthesis,*
- (ii) *every closed invariant subspace of D is the closed linear span of $\{z^n\colon\ n \in N\}$ where N is an arbitrary set of nonnegative integers,*

(iii) *for every function* $f(z) \equiv \sum_{n=0}^{\infty} a_n z^n$ *in* $\mathcal{H}(\mathbf{C})$, $\operatorname{span}\{D^j f\colon\ j \geqslant 0\} = \operatorname{span}\{z^r\colon\ a_r \neq 0\}$,

(iv) *every entire function* $f(z) \equiv \sum_{n=0}^{\infty} a_n z^n$ *with* $a_n \neq 0$ *for all* $n \geqslant 0$ *is cyclic for* D,

(v) *there do not exist sequences* $\{a_n\}$ *and* $\{l_n\}$ *of complex numbers with* $a_n \neq 0$ *for all* $n \geqslant 0$, $\limsup |a_n|^{1/n} = 0$, $\sup |l_n|^{1/n} < \infty$, *and* $\{l_n\}$, *not identically zero, such that* $0 \equiv \sum_{n=0}^{\infty} a_n l_n \lambda_n^k$ *for all* $k \geqslant 0$, *and*

(vi) *there does not exist a sequence* $\{w_n\}$ *of complex numbers, not identically zero, for which* $\limsup |w_n|^{1/n} = 0$ *and* $0 \equiv \sum_{n=0}^{\infty} w_n \lambda_n^k$ *for all* $k \geqslant 0$.

If, in addition, $\{\lambda_n/n\colon\ n \geqslant 1\}$ *is bounded, then* $\sum_{n=0}^{\infty} d_n e^{\lambda_n z}$ *is entire whenever* $\limsup |d_n|^{1/n} = 0$ *and conditions* (i)–(vi) *are equivalent to*

(vii) *there does not exist a sequence* $\{w_n\}$ *of complex numbers, not identically zero, for which* $\limsup |w_n|^{1/n} = 0$ *and* $0 \equiv \sum_{n=0}^{\infty} w_n e^{\lambda_n z}$ *for all* z *in* $\mathbf{C}$.

Proof. The equivalence of conditions (i) and (ii) was demonstrated in the remarks of the preceding Lemma 4.

(ii) $\Rightarrow$ (iii). Let $f(z) \equiv \sum_{n=0}^{\infty} a_n z^n$ be any function in $\mathcal{H}(\mathbf{C})$. Then $\operatorname{span}\{D^j f\colon\ j \geqslant 0\} \subseteq \operatorname{span}\{z^r\colon\ a_r \neq 0\} \equiv \mathcal{M}$. However by (ii), $\mathcal{M}$ is the smallest closed invariant subspace for D containing f. Hence $\mathcal{M} \subseteq \operatorname{span}\{D^j f\colon\ j \geqslant 0\}$ and so equality holds.

(iii) $\Rightarrow$ (iv). Let $f(z) \equiv \sum_{n=0}^{\infty} a_n z^n$ be any function in $\mathcal{H}(\mathbf{C})$ for which $a_n \neq 0$ for all $n \geqslant 0$. By condition (iii), $\operatorname{span}\{D^j f\colon\ j \geqslant 0\} = \operatorname{span}\{z^n\colon\ n \geqslant 0\} = \mathcal{H}(\mathbf{C})$. That is, f is cyclic for D.

(iv) $\Rightarrow$ (v). By means of contradiction, assume that there exist sequences $\{a_n\}$ and $\{l_n\}$ with the asserted properties. Then $f(z) \equiv \sum_{n=0}^{\infty} a_n z^n$ is an entire function which by (iv) is cyclic for D. Hence the closed linear span of the vectors $D^k(f) = \sum_{n=0}^{\infty} a_n \lambda_n^k z^n$ is all of $\mathcal{H}(\mathbf{C})$. Since $\sup |l_n|^{1/n} < \infty$, we have that $L(\sum_{n=0}^{\infty} c_n z^n) \equiv \sum_{n=0}^{\infty} l_n c_n$ defines a functional on $\mathcal{H}(\mathbf{C})$. Since $0 \equiv \sum_{n=0}^{\infty} l_n a_n \lambda_n^k = L(D^k(f))$ for all $k \geqslant 0$ and $\operatorname{span}\{D^k(f)\} = \mathcal{H}(\mathbf{C})$, we have that L is the zero functional. That is, $l_n \equiv 0$ for all $n \geqslant 0$, a contradiction.

(v) $\Rightarrow$ (ii). Let $\mathcal{M}$ be an arbitrary closed invariant subspace for D and define K to be the set of nonnegative integers k for which there exists a function $\sum_{n=0}^{\infty} a_n z^n$ in $\mathcal{M}$ with $a_k \neq 0$. Clearly $\mathcal{M} \subseteq \mathcal{M}_0 \equiv \operatorname{span}\{z^k\colon\ k \in K\}$. By means of contradiction, assume that $\mathcal{M} \neq \mathcal{M}_0$. Let h be any function in $\mathcal{M}_0$ which is not in $\mathcal{M}$. So there exists a nonzero functional L annihilating $\mathcal{M}$ for which $L(h) \neq 0$ (see Rudin [17] or Iyer [8]). Since L is a functional on $\mathcal{H}(\mathbf{C})$, there exists a sequence $\{l_n\}$ of complex numbers with $\sup |l_n|^{1/n} < \infty$ and $L(\sum_{n=0}^{\infty} c_n z^n) = \sum_{n=0}^{\infty} l_n c_n$ for all $\sum_{n=0}^{\infty} c_n z^n$ in $\mathcal{H}(\mathbf{C})$. By Lemma 4, there exists a function $g(z) \equiv \sum_{k \in K} a_k z^k$ with $a_k \neq 0$ for all k in K. Define $\tilde{a}_n \equiv a_n$ whenever n is in K and $\tilde{a}_n \equiv 1/n^n$ whenever n is not in K. Then $\tilde{a}_n \neq 0$ for all $n \geqslant 0$. Since g is in $\mathcal{H}(\mathbf{C})$, we have that $\limsup |a_n|^{1/n} = 0$ and so $\limsup |\tilde{a}_n|^{1/n} = 0$. Define $\tilde{l}_n \equiv l_n$ if n is in K and $\tilde{l}_n \equiv 0$ if n is not in K. Then $\sup |\tilde{l}_n|^{1/n} < \infty$ and so $\tilde{L}(\sum_{n=0}^{\infty} c_n z^n) \equiv \sum_{n=0}^{\infty} \tilde{l}_n c_n$ defines a continuous linear functional on $\mathcal{H}(\mathbf{C})$. Moreover, the sequence $\{\tilde{l}_n\}$ is not identically zero since $\tilde{L}(h) = L(h) \neq 0$. Since g is in $\mathcal{M}$, we have that $0 \equiv L(D^j g) = \sum_{k \in K} l_k a_k \lambda_k^j = \sum_{n=0}^{\infty} \tilde{l}_n \tilde{a}_n \lambda_n^j$ for all $j \geqslant 0$, contradicting (v).

(v) $\Leftrightarrow$ (vi). If $\{a_n\}$ and $\{l_n\}$ are sequences satisfying the properties of condition (v), then $w_n \equiv a_n l_n$ is a sequence satisfying the properties of condition (vi). Conversely, if $\{w_n\}$ is any sequence satisfying the properties of condition (vi), then the sequences $\{a_n\}$ and $\{l_n\}$ satisfy the properties of condition (v) where we define l_n to be 1 and a_n to be w_n if $w_n \neq 0$, and we define l_n to be 0 and a_n to be $1/n^n$ if $w_n = 0$.

(vi) $\Leftrightarrow$ (vii). If $\{\lambda_n/n\colon\ n \geqslant 1\}$ is bounded and $\limsup |a_n|^{1/n} = 0$, then the series $\sum_{n=0}^{\infty} a_n e^{\lambda_n z}$ converges absolutely uniformly on every open ball $B(0, R)$ in the complex plane by the Root Test. Hence the function $g(z) \equiv \sum_{n=0}^{\infty} a_n e^{\lambda_n z}$ is entire whenever $\limsup |a_n|^{1/n} = 0$. Moreover, in this case, $g^{(k)}(0) = \sum_{n=0}^{\infty} a_n \lambda_n^k$. Hence $\sum_{n=0}^{\infty} w_n e^{z\lambda_n} \equiv 0$ for all z in $\mathbf{C}$ if and only if $0 \equiv \sum_{n=0}^{\infty} w_n \lambda_n^k$ for all $k \geqslant 0$ whenever $\limsup |w_n|^{1/n} = 0$. $\square$

Regarding condition (vi) of Proposition 5, in 1921 Wolff [25] gave the first example of a sequence $\{w_n\}$ of complex numbers, not all zero, and a sequence $\{\lambda_n\}$ of distinct complex numbers for which $0 \equiv \sum_{n=0}^{\infty} w_n \lambda_n^k$ for all $k \geqslant 0$. In Wolff's example, the sequence $\{\lambda_n\}$ is bounded (and so $\limsup |\lambda_n|^{1/n} < \infty$) and $\{w_n\}$ is in ℓ^1. In 1952, Wermer showed that the condition $0 \equiv \sum_{n=0}^{\infty} w_n \lambda_n^k$ for all $k \geqslant 0$ is equivalent to the operator D on a separable complex Hilbert space $\mathcal{H}$ diagonalizable with respect to an orthonormal basis $\{e_n\}$ for $\mathcal{H}$ and satisfying $De_n = \lambda_n e_n$ for all $n \geqslant 0$ failing to admit spectral synthesis (see [24, Theorem 1, p. 270]). In fact, much more is known. The following lemma is an analogue of Proposition 5 for diagonalizable operators on a Hilbert space which helps illustrate the differences between the study of spectral synthesis for diagonal operators on the space $\mathcal{H}(\mathbf{C})$ of entire functions and diagonalizable operators on a separable complex Hilbert space (see Wermer [24, Theorem 1, p. 270], Sarason [18,19], Nikol'skiĭ [14, pp. 106–107], Nikol'skiĭ [15, p. 141], Brown et al. [3, Theorem 3, p. 167], and Sibilev [23, Proposition 1, Corollary 1, and Proposition 2]). Condition (iii) of Lemma 6 also makes a connection between the study of invariant subspaces of diagonalizable operators on a separable complex Hilbert space and the study of Wolff–Denjoy series of the form $\sum_{n=0}^{\infty} w_n/(z - \lambda_n)$.

Lemma 6. *Let $\mathcal{H}$ be a separable complex Hilbert space and let D be any bounded linear operator of $\mathcal{H}$ for which there exists an orthonormal basis $\{e_n\}$ for $\mathcal{H}$ and a sequence $\{\lambda_n\}$ of complex numbers for which $De_n = \lambda_n e_n$ for all $n \geqslant 0$. Then $\{\lambda_n\}$ is bounded. Moreover, D is cyclic if and only if $\lambda_m \neq \lambda_n$ for all $m \neq n$, and in this case, the following are equivalent*:

(i) *D admits spectral synthesis,*
(ii) *there does not exists a sequence $\{w_n\}$ of complex numbers in ℓ^1, not all zero, for which $0 \equiv \sum_{n=0}^{\infty} w_n \lambda_n^k$ for all $k \geqslant 0$,*
(iii) *there does not exists a sequence $\{w_n\}$ of complex numbers in ℓ^1, not all zero, for which the Wolff–Denjoy series $\sum_{n=0}^{\infty} \frac{w_n}{z-\lambda_n} \equiv 0$ for all z with $|z| > \sup |\lambda_n|$,*
(iv) *there does not exists a sequence $\{w_n\}$ of complex numbers in ℓ^1, not all zero, for which the complex measure $\mu \equiv \sum_{n=0}^{\infty} w_n \delta_{\{\lambda_n\}}$ consisting of point masses at the λ_n with weights w_n annihilates the polynomials,*
(v) *there does not exists a sequence $\{w_n\}$ of complex numbers in ℓ^1, not all zero, for which the exponential series $\sum_{n=0}^{\infty} w_n e^{\lambda_n z} \equiv 0$ on the complex plane,*
(vi) *every closed invariant subspace of D is invariant for the adjoint D^* of D, and*

(vii) *the adjoint D^* of D is in the weakly closed algebra generated by the identity operator and D.*

If, in addition, the λ_n lie inside a Jordan region G and accumulate only on the boundary of G, then conditions (i)–(vii) *are equivalent to*

(viii) $\sup\{|f(z)|\colon z \in G\} = \sup\{|f(\lambda_n)|\colon n \geqslant 0\}$.

The study of Wolff–Denjoy series $\sum_{n=0}^{\infty} w_n/(z-\lambda_n)$ has a long and rich history. Of particular interest has been conditions for an analytic function to be representable as a Wolff–Denjoy series, and conditions for such a representation, if one exists, to be unique. Borel, Beurling, and Carleman all gave sufficient conditions for the representation of an analytic function as a Wolff–Denjoy series to be unique in terms of the rate of decay of the coefficients in the series. Sibilev, in 1995, gave a definitive uniqueness theorem of this type (see Sibilev [23]). Wolff–Denjoy series have also been studied extensively by Poincaré, Wolff, Borel, Carleman, and Beurling, among others, mainly in connection with quasianalyticity and analytic continuation (see the recent monograph of Ross and Shapiro [16]).

Wolff's 1921 example has been extended to sequences $\{\lambda_n\}$ of distinct complex numbers which are unbounded. For instance, in 1936, Natanson showed that there exists a sequence $\{w_n\}$ of complex numbers for which $\sum_{n=0}^{\infty} |w_n||\lambda_n|^k < \infty$ and $0 \equiv \sum_{n=0}^{\infty} w_n \lambda_n^k$ for all $k \geqslant 0$ in the special case $\lambda_n = n$ for all $n \geqslant 0$ (see 5.7.8c(v) on p. 128 of Nikol'skiǐ [14]). In 1959, Makarov generalized Natanson's example to include any sequence $\{\lambda_n\}$ of complex numbers for which $|\lambda_n| \to \infty$ (see 5.7.8c(vi) on p. 128 of Nikol'skiǐ [14]). We will see, however, in Lemma 9 that the diagonal operator on $\mathcal{H}(\mathbf{C})$ having eigenvalues $\lambda_n = n$ admits spectral synthesis. Hence the coefficients $\{w_n\}$ in Natanson's example are such that $\limsup |w_n|^{1/n} > 0$ by Proposition 5(vi).

It remains an open question as to whether or not every cyclic diagonal operator on $\mathcal{H}(\mathbf{C})$ admits spectral synthesis. That is, it is not known if there exists a sequence $\{w_n\}$ of complex numbers, not all zero, with $\limsup |w_n|^{1/n} = 0$ and a sequence $\{\lambda_n\}$ of distinct complex numbers for which $\limsup |\lambda_n|^{1/n} < \infty$ and $0 \equiv \sum_{n=0}^{\infty} w_n \lambda_n^k$ for all $k \geqslant 0$.

We show, however, that every cyclic diagonal operator D on $\mathcal{H}(\mathbf{C})$ whose eigenvalues $\{\lambda_n\}$ are bounded admits spectral synthesis using the uniqueness result of Sibilev mentioned earlier.

Lemma 7. *Every cyclic diagonal operator D on $\mathcal{H}(\mathbf{C})$ whose eigenvalues $\{\lambda_n\}$ are bounded admits spectral synthesis.*

Proof. By means of contradiction, assume that D is a cyclic diagonal operator on $\mathcal{H}(\mathbf{C})$ whose eigenvalues $\{\lambda_n\}$ are bounded but which fails spectral synthesis. Without loss of generality, we may assume that $|\lambda_n| < 1$ for all $n \geqslant 0$. By Proposition 5(vii), there exists a sequence $\{w_n\}$ of complex numbers, not identically zero, for which $\limsup |w_n|^{1/n} = 0$ and $0 \equiv \sum_{n=0}^{\infty} w_n e^{\lambda_n z}$ for all z in $\mathbf{C}$. Since $\limsup |w_n|^{1/n} = 0$, we have that $|w_n| \leqslant 1/2^n$ for all n sufficiently large and so there exists a constant c for which $|w_n| \leqslant c/2^n$ for all $n \geqslant 0$. In particular, $\{w_n\}$ is in ℓ^1. Hence by Proposition 2 of Sibilev [23, p. 147], $0 \equiv$

$\sum_{n=0}^{\infty} w_n/(z-\lambda_n)$ whenever $|z|>1$. Since $\sum\{\ln(c/2^n)\}/n^2=-\infty$, we have that $w_n\equiv 0$ for all $n\geqslant 0$ by the theorem on p. 146 of Sibilev [23], a contradiction. □

It follows from Lemma 7 that there exist cyclic diagonal operators on $\mathcal{H}(\mathbf{C})$ admitting spectral synthesis the closure of whose eigenvalues $\{\lambda_n\}$ have nonempty interior. This is in contrast to the case for diagonalizable operators on a separable complex Hilbert space (see Scroggs [21, Corollary 3.1, p. 104]).

The following result yields examples of cyclic diagonal operators on $\mathcal{H}(\mathbf{C})$ admitting spectral synthesis whose eigenvalues may be unbounded.

Theorem 8. *Let D be any diagonal operator on $\mathcal{H}(\mathbf{C})$ having eigenvalues $\{\lambda_n\}$. If $\{\lambda_n/n\colon n\geqslant 1\}$ is bounded and the real parts of the λ_n are strictly increasing, then D admits spectral synthesis.*

Proof. By means of contradiction, assume that D fails to admit spectral synthesis. So by condition (vii) of Proposition 5, there exists a sequence $\{w_n\}$ of complex numbers, not all zero, for which $\limsup|w_n|^{1/n}=0$ and $0\equiv\sum_{n=0}^{\infty}w_ne^{\lambda_n z}$ for all z in $\mathbf{C}$. Define N to be the smallest nonnegative integer for which $w_N\neq 0$. Then $0=w_1=w_2=\cdots=w_{N-1}$, and so $w_Ne^{\lambda_N z}=-\sum_{n=N+1}^{\infty}w_ne^{\lambda_n z}$ for all complex numbers z. Upon dividing by $e^{\lambda_N z}$, we see that $0\neq|w_N|\leqslant\sum_{n=N+1}^{\infty}|w_ne^{(\lambda_n-\lambda_N)z}|\leqslant\sum_{n=N+1}^{\infty}|w_n|e^{\mathrm{Re}(\lambda_n-\lambda_N)z}$ for all $z<0$ since the real parts of λ_n are strictly increasing. Since $\limsup|w_n|^{1/n}=0$, we have that $\{w_n\}$ is in ℓ^1 and so we obtain a contradiction upon letting z tend to $-\infty$. □

It follows from Theorem 8 that the diagonal operator D on $\mathcal{H}(\mathbf{C})$ having eigenvalues $\lambda_n\equiv n$ admits spectral synthesis.

Lemma 9. *The diagonal operator D on $\mathcal{H}(\mathbf{C})$ having eigenvalues $\lambda_n=n$ admits spectral synthesis.*

4. Common cyclic vectors

We say that a vector x in a complete metrizable topological vector space $\mathcal{X}$ is a *common cyclic vector* for a set $\mathcal{D}$ of cyclic operators on $\mathcal{X}$ if x is a cyclic vector for each operator D in $\mathcal{D}$. Herrero has shown that a cyclic operator on a Banach space has a dense set of cyclic vectors if and only if the point spectrum of its adjoint has empty interior (see [6, Theorem 1, p. 918]). Moreover, Shields has shown that the set of cyclic vectors of an operator on a Banach space is a $\mathcal{G}_\delta$ set (see [22, Proposition 40, p. 411]). Hence by the Baire Category Theorem any countable collection of cyclic operators on a Banach space the point spectra of all of whose adjoints have empty interior has a dense set of common cyclic vectors.

In this section, we show that the uncountable collection of cyclic diagonal operators on $\mathcal{H}(\mathbf{C})$ each of whose eigenvalues are separated (in a sense made precise below) has a dense set of common cyclic vectors.

Theorem 10. *Let $\mathcal{D}_0$ denote the collection of cyclic diagonal operators on $\mathcal{H}(\mathbf{C})$ each of whose set of eigenvalues $\{\lambda_n\}$ is such that* $\inf\{|\lambda_i - \lambda_j|: i \neq j\} > 0$. *Then $\mathcal{D}_0$ has a dense set of common cyclic vectors.*

Proof. We show that the set $\mathcal{C}$ of entire functions $\sum_{k=0}^{\infty} a_k z^k$ for which there exists a constant α with $0 < |a_r| \leqslant \alpha / r^{r^r}$ for all $r \geqslant 0$ is a dense set of common cyclic vectors for $\mathcal{D}_0$. To this end, let $f_0(z) \equiv \sum_{k=0} a_k z^k$ be an arbitrary function in $\mathcal{C}$ and let D be an arbitrary diagonal operator in $\mathcal{D}_0$. We show that f_0 is cyclic for D. As in the proof of Proposition 3, it suffices to show that $a_k z^k$ is in the closure of the orbit of f_0 under D for all $k \geqslant 0$. To this end, let L be an arbitrary functional on $\mathcal{H}(\mathbf{C})$. So there exists a sequence $\{l_n\}$ of complex numbers for which $B \equiv \sup |l_n|^{1/n} < \infty$ and $L(\sum_{k=0}^{\infty} c_k z^k) = \sum_{k=0}^{\infty} l_k c_k$ for all functions $\sum_{k=0}^{\infty} c_k z^k$ in $\mathcal{H}(\mathbf{C})$. Without loss of generality, $B \geqslant 1$.

For each positive integer n and each k in $\{0, 1, \dots, n\}$, the polynomial

$$p_{n,k}(z) \equiv \prod_{i=0,\, i \neq k}^{n} \frac{z - \lambda_i}{\lambda_k - \lambda_i}$$

is well-defined since D, being cyclic, has distinct eigenvalues. We have that $p_{n,k}(\lambda_r) = 0$ for all $r \leqslant n$ with $r \neq k$, and that $p_{n,k}(\lambda_k) = 1$ for all n with $k \leqslant n$. Since D is continuous, $K \equiv \limsup |\lambda_r|^{1/r} < \infty$ and so $|\lambda_r| \leqslant (K+1)^r$ for all r sufficiently large. Hence there exists a constant $c \geqslant 1$ for which $|\lambda_r| \leqslant c(K+1)^r$ for all $r \geqslant 0$. Since D is in $\mathcal{D}_0$, $\delta \equiv \min\{1, \inf\{|\lambda_k - \lambda_i|: i \neq k\}\} > 0$. It follows that $|p_{n,k}(\lambda_r)| \leqslant \{2c(K+1)^r/\delta\}^n$ for all $r > n$. For any integer n greater than $4cB(K+1)/\delta$, we have that $B(K+1)^n/n^n \leqslant \delta/(4c)$, and so

$$\begin{aligned}
\bigl|L\bigl(p_{n,k}(D) f_0 - a_k z^k\bigr)\bigr| &= \left|\sum_{r>n} p_{n,k}(\lambda_r) a_r l_r\right| \leqslant \sum_{r>n} \left\{\frac{2c(K+1)^r}{\delta}\right\}^n \frac{\alpha}{r^{r^r}} B^r \\
&\leqslant \alpha \left\{\frac{2c}{\delta}\right\}^n \sum_{r>n} \left\{\frac{B(K+1)^n}{n^n}\right\}^r \leqslant \alpha \left\{\frac{2c}{\delta}\right\}^n \sum_{r>n} \left\{\frac{\delta}{4c}\right\}^{rn} \\
&\leqslant \alpha \left\{\frac{2c}{\delta}\right\}^n \sum_{r>n} \left\{\frac{\delta}{4c}\right\}^r = \frac{\alpha\delta}{2^n(4c - \delta)}
\end{aligned}$$

which tends to zero as n tends to infinity. Since L is an arbitrary functional on $\mathcal{H}(\mathbf{C})$, $a_k z^k$ is in the weak closure of the linear span of the orbit of f_0 under D. Since f_0 is in $\mathcal{C}$, $a_k \neq 0$ and so z^k is in the weak closure of the linear span of the orbit of f_0 under D for all $k \geqslant 0$. Since the weak closure of the linear span of the orbit of f_0 under D and the closure of the linear span of the orbit of f_0 under D in $\mathcal{H}(\mathbf{C})$ coincide and the monomials have dense linear span in $\mathcal{H}(\mathbf{C})$, it follows that f_0 is cyclic for D. An argument similar to the one given in the proof of Proposition 3 shows that $\mathcal{C}$ is dense in $\mathcal{H}(\mathbf{C})$. The result follows. □

The rate of decay of the coefficients defining the collection $\mathcal{C}$ of entire functions in the preceding proof may be improved significantly.

It is not known if the set of all cyclic diagonal operators on $\mathcal{H}(\mathbf{C})$ has a common cyclic vector.

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